BC-TYPE INTERPOLATION MACDONALD POLYNOMIALS AND BINOMIAL FORMULA FOR KOORNWINDER POLYNOMIALS

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Abstract. We consider 3-parametric polynomials $P^*_\mu(x;q,t,s)$ which replace the A_n-series interpolation Macdonald polynomials $P^*_\mu(x;q,t)$ for the BC_n-type root system. For these polynomials we prove an integral representation, a combinatorial formula, Pieri rules, Cauchy identity, and we also show that they do not satisfy any rational q-difference equation. As $s \to \infty$ the polynomials $P^*_\mu(x;q,t,s)$ become $P^*_\mu(x;q,t)$. We also prove a binomial formula for 6-parametric Koornwinder polynomials.

Contents

0. Introduction

The higher Capelli identities [N,Ok1-2] describe two faces of a distinguished linear basis in the center of $\mathcal{U}(\mathfrak{gl}(n))$. The elements of this basis, which are called *quantum immanants*, have also a third and a very important face. It is their Harish-Chandra image called *shifted Schur functions*, or s^*–functions for short. The s^*–functions have remarkably many applications, see [OO] and, for example, also [KOO]. In particular, they were used in the proof of the higher Capelli identities given in [Ok1].

Since the definition of the s^*–functions involves only the Weyl group and the half-sum of positive roots ρ, they can be generalized in a quite far reaching way. This was done in the A_n case by F. Knop, G. Olshanski, S. Sahi, and the author in [KS,Kn,S,Ok3,OO3] and other papers, see References. This theory is remarkably parallel to the theory of ordinary Macdonald polynomials $P_\mu(x;q,t)$ for the root system A_n. In some aspects, such as e.g. duality [Ok3], §6, or binomial formula [Ok4], it looks even more natural.

The definition of the polynomials $P^*_\mu(x;q,t)$, which generalize shifted Schur functions in the way that

$$s^*_\mu(x) = \lim_{q\to 1} \frac{P^*_\mu(q^x;q,q)}{(q-1)^{|\mu|}}\,,$$

where

$$q^x = (q^{x_1},\dots,q^{x_n})$$

and $|\mu|$ stands for the number of squares in μ, is the following. By definition, the polynomial $P^*_\mu(x;q,t)$ is the unique, up to a scalar, polynomial of degree $|\mu|$ that is symmetric in variables

$$x_1t^{n-1}, x_2t^{n-2},\dots,x_n\,, \tag{0.1}$$

and vanishes at the points

$$P^*_\mu(q^\lambda;q,t) = 0\,, \quad \mu \not\subset \lambda\,,$$

where $\lambda = (\lambda_1,\dots,\lambda_n)$ is a partition. We call these polynomials *interpolation Macdonald polynomials*. They are also known as shifted or inhomogeneous Macdonald polynomials.

The polynomials $P^*_\mu(x;q,t)$ are the Harish-Chandra image

$$\mathfrak{D} \xrightarrow{\text{Harish-Chandra isomorphism}} \mathbb{C}(q,t)[x_1t^{n-1}, x_2t^{n-2},\dots,x_n]^{S(n)}$$

of some distinguished linear basis of the commutative algebra $\mathfrak{D}$ of Macdonald q-difference operators. Recall that the algebra $\mathfrak{D}$ acts diagonally in the basis of Macdonald polynomials and the Harish-Chandra homomorphism associates to any element of $\mathfrak{D}$ its eigenvalues on the Macdonald polynomials $P_\mu(x;q,t)$ viewed as a function of q^μ.

Informally, by analogy with [Ok1], one can think of the polynomials $P^*_\mu(x;q,t)$ as of the Harish-Chandra image of certain Capelli-type Laplace operators on on some fictitious "quantum symmetric space" with an A_n-type restricted root system. In the three cases

$$t = q^\theta\,, \quad \theta = \frac{1}{2}, 1, 2$$

there exist certain quantum symmetric spaces (see [No2]) which are q-analogs of

$$\begin{aligned} &U(n)/O(n)\,, && \theta = 1/2\,,\\ &U(n)\,, && \theta = 1\,,\\ &U(2n)/Sp(2n)\,, && \theta = 2\,. \end{aligned}$$

Remark, that from the definition it is not obvious at all that the polynomials $P^*_\mu(x;q,t)$ are anyhow related to ordinary Macdonald polynomials. There is, therefore, a natural question if any such relation exists for other root systems. The root system of maximal interest for applications is the non-reduced root system of type BC_n, where we have even more "live" symmetric spaces than in the A_n case. In [OO2], we considered the analogs of shifted Schur functions for other classical groups.

In this paper we study the following natural BC_n-type analogs of the interpolation Macdonald polynomials $P^*_\mu(x;q,t)$. By definition, the polynomial

$$P^*_\mu(x;q,t,s) \in \mathbb{C}(q,t,s)[x_1^{\pm 1},\dots,x_n^{\pm 1}]$$

is the unique, up to a scalar, polynomial of degree $|\mu|$ which is symmetric in variables (0.1) and also invariant under the transformations

$$x_i t^{n-i} s \mapsto \frac{1}{x_i t^{n-i} s}\,,$$

such that

$$P^*_\mu(q^\lambda;q,t,s) = 0 \quad \text{if} \quad \mu \not\subset \lambda$$

for any partition λ. The vector $\rho = (\rho_1,\dots,\rho_n)$ such that

$$q^\rho = (t^{n-1}s,\dots,ts,s)$$

plays the role of the half-sum of positive roots taken with multiplicities.

Our analysis shows that this BC_n-type case has some new and unexpected features.

We prove that the polynomials $P^*_\mu(x;q,t,s)$ are actually very close to the polynomials $P^*_\mu(x;q,t)$. In particular, as $s \to \infty$ all negative powers of variables disappear and

$$P^*_\mu(x;q,t,s) \to P^*_\mu(x;q,t), \quad s \to \infty\,. \tag{0.2}$$

Moreover, the new parameter s can be inserted into the two explicit formulas for $P^*_\mu(x;q,t)$ obtained in [Ok3], namely, the *integral representation* and the *combinatorial formula*, in such a way that (0.2) becomes evident. This is done in sections 3–5, where the proofs are suitable modifications of the proofs from [Ok3]. There is, however, also a major difference between the polynomials $P^*_\mu(x;q,t,s)$ and $P^*_\mu(x;q,t)$, see below.

Recall that in the s^*–functions case the combinatorial formula and the integral representation, which was called in that context the *coherence property*, have a clear interpretation in terms of the higher Capelli identities. The combinatorial formula reflects the formula (3.24) in [Ok1] for the LHS of the higher Capelli identities, see [Ok1], §3.7. The coherence property describes the action of the averaging operator

$$\text{center of } \mathcal{U}(\mathfrak{gl}(n)) \xrightarrow{\text{averaging}} \text{center of } \mathcal{U}(\mathfrak{gl}(N))\,, \quad n < N$$

in the basis of quantum immanants. This action can be easily computed either by explicit averaging of the differential operator in the RHS of the higher Capelli identities, see [Ok1], §5.1, or by using the vanishing property, see [OO], §10.

The integral representation (3.1) is in fact a natural analog of the classical Weyl character formula. The denominator $D(x^*;q,t)$ in (3.1) is a straightforward generalization of the product in the denominator of the Weyl formula. The alternating sum in the numerator of the Weyl formula turns into a multivariate q-integral which

is, by definition, alternating sum of contributions of lower and upper bounds of integration. One proves that the numerator is divisible by the denominator using, essentially, the fact that any q-integral

$$\int_v^u z^l \, d_q z \,, \quad l \neq -1 \,,$$

is divisible by $u - v$ in $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, see Appendix 1.

On the other hand, the relation of the polynomials $P^*_\mu(x; q, t, s)$ to the Macdonald polynomials for the BC_n root system is only an indirect one, via the *binomial theorem*. In fact, it is natural to work with *Koornwinder polynomials* [K], which generalize Macdonald polynomials for classical root systems and have as many as 6 parameters. We show that using the polynomials $P^*_\mu(x; q, t, s)$ one obtains a binomial formula for Koornwinder polynomials just along the same lines as in [Ok4]. The only property of the polynomials $P^*_\mu(x; q, t, s)$ needed in the proof of the binomial formula, as an abstract identity of symmetric polynomials, is the identification (4.1) of the top homogeneous term of $P^*_\mu(x; q, t, s)$.

As an application of this binomial formula, or, more precisely, of its classical limit, one obtains a BC analog of the results of [OO4]. The details will be given in a forthcoming paper by G. Olshanski and the author.

We conclude this introduction with remarks on some open problems and also some *no go* results.

By the binomial theorem, see [OO3], §6, and [Ok4], the integral representation of $P^*_\mu(x; q, t)$ reflects the *branching rule* for the ordinary A_n-type Macdonald polynomials. It is likely that the integral representation of the polynomials $P^*_\mu(x; q, t, s)$ reflects branching rules for the Koornwinder polynomials. The

$$s \mapsto s t^{1/2}$$

shift in the integral representation (3.1) suggests that the branching rules for Koornwinder polynomials must have two natural steps just like the branching rules for characters of orthogonal and symplectic series.

The combinatorial formula for the A_n-type polynomials $P^*_\mu(x; q, t)$ and the integral representation of $P^*_\mu(x; q, t)$ are equivalent to each other by virtue of the following symmetry, see [Ok3], formula (2.1),

$$\frac{P^*_\mu(aq^{-\lambda_n}, \dots, aq^{-\lambda_1}; q, t)}{P^*_\mu(a, \dots, a; q, t)} = \frac{P^*_\lambda(aq^{-\mu_n}, \dots, aq^{-\mu_1}; q, t)}{P^*_\lambda(a, \dots, a; q, t)} \tag{0.3}$$

between the argument x and the label μ of $P^*_\mu(x; q, t)$. Here a is an arbitrary number. As $a \to \infty$ the symmetry (0.3) becomes the symmetry for ordinary Macdonald polynomials.

It looks like no such symmetry connects the integral representation (3.1) to the combinatorial formula (5.3) for the BC_n-type interpolation polynomials. First of all, the symmetry (0.3) uses the automorphism of the weight lattice

$$\lambda \mapsto -w_0(\lambda) \,,$$

which is trivial in the BC_n case. Here w_0 stands for the longest element of the Weyl group. Another obstruction to symmetry is that the weight function $\psi_{\lambda/\mu}$ in the combinatorial formula, which is the same $\psi_{\lambda/\mu}$ as in [M], VI.6.24 and Example VI.6.2.(b),

$$\psi_{\lambda/\mu} = \prod_{i\le j\le \ell(\mu)} \frac{(q^{\mu_i-\mu_j}t^{j-i+1})_\infty}{(q^{\mu_i-\mu_j+1}t^{j-i})_\infty} \frac{(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1})_\infty}{(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i})_\infty} \times \frac{(q^{\lambda_i-\mu_j+1}t^{j-i})_\infty}{(q^{\lambda_i-\mu_j}t^{j-i+1})_\infty} \frac{(q^{\mu_i-\lambda_{j+1}+1}t^{j-i})_\infty}{(q^{\mu_i-\lambda_{j+1}}t^{j-i+1})_\infty} \tag{0.4}$$

is, unlike the weight function in the integral representation (3.1), not invariant under transformations of the form

$$q^{\lambda_i}t^{n-i}s \mapsto \frac{1}{q^{\lambda_i}t^{n-i}s}\,.$$

This problem is closely related to the following important difference between the polynomials A_n and BC_n case. The polynomials $P^*_\mu(x;q,t,s)$ *do not* satisfy any q-difference equations with rational coefficients, see Appendix 2. Recall that there are very useful q-difference equations for $P^*_\mu(x;q,t)$ discovered in [K,S2]. Explicit formulas for higher order difference equations can be found in [Ok4], §3. It seems to be a very interesting and important problem to find a geometric or representation-theoretic meaning of the Knop-Sahi difference equation and to understand why they do not have obvious BC-analogs.

Finally, it is natural to ask if one can add some more parameters to the 3-parametric polynomials $P^*_\mu(x;q,t,s)$ and still preserve some of their properties. It has been shown in [Ok5] that the property

$$P^*_\mu(q^\lambda;q,t,s) = 0\,, \quad \mu \not\subset \lambda\,,$$

which requires the polynomials $P^*_\mu(x;q,t,s)$ to vanish at an infinite set of points q^λ, $\mu\not\subset\lambda$, characterizes the polynomials $P^*_\mu(x;q,t,s)$ inside a very general class of interpolation polynomials. It follows from this characterization, see [Ok5], that neither integral representation nor combinatorial formula exist for any more general interpolation polynomials in that class.

1. Definition and normalization

Let q,t,s be three parameters. We shall assume that

$$q^i t^j s^k \neq 1 \quad \text{for all} \quad i,j,k\in\mathbb{Z}_+\,.$$

Set

$$\Bbbk = \mathbb{C}(q,t,s)\,.$$

We shall assume that

$$q,t\in\mathbb{C} \quad \text{and} \quad |q|,|t|<1$$

wherever convergence is involved. We shall also need the square roots $q^{1/2}$ and $t^{1/2}$. Consider the BC-type Weyl group

$$W_n = S(n) \ltimes \mathbb{Z}_2^n$$

and its standard action on $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Denote by

$$\Lambda_{t,s} \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

the subalgebra of polynomials that are W-invariant in new variables

$$x_i^* = x_i t^{n-i} s\,, \quad i = 1, \dots, n\,. \tag{1.1}$$

Note that for any $f \in \Lambda_{t,s}$

$$\deg f \geq 0$$

and $\deg f = 0$ only for constant polynomials. Here $\deg f$ denotes the usual degree, that is the maximal total degree in $x_1, \dots, x_n$ of all monomials of f.

Definition 1.1. Let μ be a partition with at most n parts. By definition,

$$P_\mu^*(x_1, \dots, x_n; q, t, s)$$

is the element of $\Lambda_{t,s}$ satisfying the following conditions:

(1) $\deg P_\mu^*(x; q, t, s) \leq |\mu|$,
(2) $P_\mu^*(q^\lambda; q, t, s) = 0$ if $\mu \not\subset \lambda$,
(3) $P_\mu^*(q^\mu; q, t, s) = H(\mu, n; q, t, s)$.

Here $H(\mu, n; q, t, s)$ is just a nonzero normalization constant which we shall specify below.

It is clear that existence of P_μ^* implies uniqueness. Below we shall give two explicit formulas for P_μ^*. It is also clear that

Proposition 1.1. *The polynomials $P_\mu^*(x; q, t, s)$, where μ ranges over partitions with at most n parts, form a $\Bbbk$-basis of the vector space $\Lambda_{t,s}$. The degree*

(1′) $\deg P_\mu^*(x; q, t, s) = |\mu|$

is exactly $|\mu|$.

For any polynomial

$$f(x) \in \Lambda_{t,s}$$

the coefficients f_μ in the expansion

$$f(x) = \sum_\mu f_\mu P_\mu^*(x; q, t, s)$$

can be found from the following non-degenerate triangular (with respect to the ordering of partitions by inclusion) system of linear equations

$$f(q^\lambda) = \sum_\mu f_\mu P_\mu^*(q^\lambda; q, t, s)\,.$$

Here λ ranges over all partitions with at most n parts.

Now we shall specify the constant $H(\mu, n; q, t, s)$. Consider the diagram

$$\mu = (\mu_1, \dots, \mu_n)$$

as the following skew diagram

$$\mu = M/\boldsymbol{\mu}\,,$$

where

$$\begin{aligned} M &= (2\mu_1, \mu_1+\mu_2, \dots, \mu_1+\mu_n, \mu_1-\mu_n, \dots, \mu_1-\mu_2, 0)\,, \\ \boldsymbol{\mu} &= (\mu_1, \mu_1, \dots, \mu_1, \mu_1-\mu_n, \dots, \mu_1-\mu_2, 0)\,. \end{aligned}$$

For the diagram

$$\mu = (4,2,1,1)$$

the diagram M looks as follows (the black squares correspond to the subdiagram $\boldsymbol{\mu}$)

$$\begin{array}{cccccccc} \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & \square & \square & \square \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & \square & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \square & & & \\ \blacksquare & \blacksquare & \blacksquare & & & & & \\ \blacksquare & \blacksquare & \blacksquare & & & & & \\ \blacksquare & \blacksquare & & & & & & \end{array}$$

Each square

$$\square \in \mu$$

has its mirror image $\boxtimes$ in the diagram $\boldsymbol{\mu}$. If $\square$ is the i-th row and j-th column of μ then $\boxtimes$ is in the i-th row and $(\mu_i - j + 1)$-st column of $\boldsymbol{\mu}$. The mirror images of four squares $\square \in \mu = (4,2,1,1)$ are given in the following picture

$$\begin{array}{cccccccc} \boxtimes & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \boxtimes & \cdot & \cdot & \square & & \\ \cdot & \cdot & \cdot & \boxtimes & \square & & & \\ \cdot & \cdot & \cdot & \boxtimes & \square & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & & & & & & \end{array}$$

Recall the following notations of Macdonald, see [M], Ch. I. Given a partition μ set

$$n(\mu) = \sum_i (i-1)\mu_i = \sum_j \mu'_j(\mu'_j - 1)/2\,.$$

Recall that for each square $\square = (i,j) \in \mu$ the numbers

$$\begin{aligned} a(\square) &= \mu_i - j, \qquad & a'(\square) &= j - 1, \\ l(\square) &= \mu'_j - i, \qquad & l'(\square) &= i - 1, \end{aligned}$$

are called arm-length, arm-colength, leg-length, and leg-colength respectively. The numbers

$$a(\boxtimes) = \mu_i + a'(\square)\,, \qquad l(\boxtimes) = l(\square) + 2(n - \mu'_j)$$

are the arm-length and leg-length of the mirror image $\boxtimes$ of $\square$ measured with respect to the big diagram M.

Definition 1.2. $H(\mu, n; q, t, s)$ is the following normalization constant

$$\frac{t^{n(\mu)-2(n-1)|\mu|}}{q^{2n(\mu')+|\mu|}s^{2|\mu|}} \prod_{\square\in\mu} \left(q^{a(\square)+1}t^{l(\square)}-1\right)\left(s^2 q^{a(\boxtimes)}t^{l(\boxtimes)}-1\right).$$

Observe that

$$\sum_{\square\in\mu} a(\boxtimes) = |\mu| + 3n(\mu'),$$
$$\sum_{\square\in\mu} l(\boxtimes) = 2(n-1)|\mu| - 3n(\mu).$$

Therefore,

$$H(\mu, n; q, t, s) \to \frac{q^{n(\mu')}}{t^{2n(\mu)}} \prod_{\square\in\mu} \left(q^{a(\square)+1}t^{l(\square)}-1\right), \quad s\to\infty,$$

which is the normalization constant for two-parameter interpolation Macdonald polynomials, see [Ok3].

Recall also that

$$\sum_{\square\in\mu} a(\square) = n(\mu'), \quad \sum_{\square\in\mu} l(\square) = n(\mu).$$

Remark that $P^*_\mu(x; q, t, s)$ depends only on s^2, not s, because the polynomial $P^*_\mu(x; q, t, -s)$ satisfies all conditions of the definition of $P^*_\mu(x; q, t, s)$.

2. Elementary properties

We have the three following elementary propositions.

Proposition 2.1. *Suppose* $\mu_n > 0$ *and put*

$$\mu - \bar{1} = (\mu_1 - 1, \dots, \mu_n - 1).$$

Then

(2.1) $P^*_\mu(x; q, t, s) =$

$$q^{|\mu-\bar{1}|}\prod_i (x_i t^{1-i} - t^{1-n})\left(1 - \frac{1}{x_i t^{n-i} s^2}\right) P^*_{\mu-\bar{1}}(x/q; q, t, sq).$$

Proof. It is clear the RHS of (2.1) satisfies all conditions of the definition 1.1 except for normalization. Evaluate it at $x = q^\mu$. We obtain

$$q^{-|\mu|+|\mu-\bar{1}|}t^{-3n(n-1)/2}s^{-2n}\times$$
$$\prod_i \left(q^{\mu_i}t^{n-i}-1\right)\left(s^2 t^{n-i}q^{\mu_i}-1\right) H(\mu-\bar{1}, n; q, t, sq),$$

which equals $H(\mu, m; q, t, s)$ because

$$\begin{aligned} n(\mu) &= n(\mu-\bar{1}) + n(n-1)/2, \\ n(\mu') &= n((\mu-\bar{1})') + |\mu-\bar{1}|, \\ |\mu| &= |\mu-\bar{1}| + n. \end{aligned}$$

This concludes the proof. $\square$

A similar computation proves the following

Proposition 2.2. *Suppose* $\mu_n = 0$. *Then*

$$P^*_\mu(x_1,\dots,x_{n-1},1;q,t,s) = P^*_\mu(x_1,\dots,x_{n-1};q,t,st)\,. \tag{2.2}$$

Proposition 2.3.

$$P^*_\mu(1/x_1,\dots,1/x_n;1/q,1/t,1/s) = s^{2|\mu|}t^{(2n-2)|\mu|}P^*_\mu(x_1,\dots,x_n;q,t,s)\,.$$

Proof. Again, it is clear that the LHS satisfies all conditions of the definition of $P^*_\mu(x;q,t,s)$ except for normalization. Compute $H(\mu,n;1/q,1/t,1/s)$. We obtain

$$\frac{q^{2n(\mu')+|\mu|}s^{2|\mu|}}{t^{n(\mu)-2(n-1)|\mu|}}\prod_{\square\in\mu}\left(q^{-a(\square)-1}t^{-l(\square)}-1\right)\left(s^{-2}q^{-a(\boxtimes)}t^{-l(\boxtimes)}-1\right),$$

which equals

$$\frac{t^{n(\mu)}}{q^{|\mu|+2n(\mu')}}\prod_{\square\in\mu}\left(q^{a(\square)+1}t^{l(\square)}-1\right)\left(s^2q^{a(\boxtimes)}t^{l(\boxtimes)}-1\right)=$$

$$= s^{2|\mu|}t^{(2n-2)|\mu|}H(\mu,n;q,t,s)\,. \quad \square$$

3. Integral representation

In this section we obtain a q-integral representation of $P^*_\mu(x;q,t,s)$, which is a way to obtain the polynomial

$$P^*_\mu(x_1,\dots,x_n;q,t,s)$$

in n variables from the polynomial

$$P^*_\mu(y_1,\dots,y_{n-1};q,t,st^{1/2})$$

in $n-1$ variable.

Basic facts about q-integrals are recalled in the Appendix. Introduce some notations; set

$$(a)_\infty = (1-a)(1-qa)(1-q^2a)\dots\,.$$

Consider the following products

$$V(x_1,\dots,x_n) = \det\left[\left(x_i^{n-j+1}-x_i^{-(n-j+1)}\right)\right]_{1\le i,j\le n},$$

$$\Pi(x_1,\dots,x_n;y_1,\dots,y_{n-1};q,t) = \prod_{i,j}\frac{(q^{1/2}(x_i)^{\pm1}(y_j)^{\pm1})_\infty}{(t^{1/2}(x_i)^{\pm1}(y_j)^{\pm1})_\infty},$$

$$D(x_1,\dots,x_n;q,t) = \prod_{i<j}((x_i+x_i^{-1})-(x_j+x_j^{-1}))\frac{(q(x_i)^{\pm1}(x_j)^{\pm1})_\infty}{(t(x_i)^{\pm1}(x_j)^{\pm1})_\infty}\,.$$

Set also

$$y_i^* = y_it^{n-i-1/2}s, \quad i=1,\dots,n-1\,.$$

We shall integrate over the domain

$$\int_{y\preccurlyeq x} d_qy = \int_{x_2}^{qx_1} d_qy_1\cdots\int_{x_n}^{qx_{n-1}} d_qy_{n-1}$$

with respect to the following beta-type measure

$$\beta_{q,t}(d_qy) = V(y^*;q,t)\,\Pi(x^*;y^*;q,t)\,\frac{d_qy_1}{y_1}\dots\frac{d_qy_{n-1}}{y_{n-1}}\,.$$

Here the variables x^* are defined in (1.1). We have

Theorem 3.1 (Integral representation). *Suppose* $\mu_n = 0$. *Then*

$$P^*_\mu(x;q,t,s) = \frac{1}{C(\mu,n)}\frac{t^{-|\mu|}}{D(x^*;q,t)}\int_{y\preccurlyeq x}\beta_{q,t}(d_q y)\,P^*_\mu(y;q,t,st^{1/2})\,. \tag{3.1}$$

Here

$$\begin{aligned} C(\mu,n) &= q^{n(n-1)/4}\prod_{i<n} B_q(\mu_i+(n-i)\theta,\theta) \\ &= q^{n(n-1)/4}(1-q)^{(n-1)}\prod_{i<n}\frac{(q)_\infty}{(t)_\infty}\frac{(q^{\mu_i}t^{n-i+1})_\infty}{(q^{\mu_i}t^{n-i})_\infty}\,, \end{aligned}$$

where $t=q^\theta$ *and* B_q *is the* q*-beta function* (A.3)*.*

This theorem together with proposition 2.1 gives a recursive formula for all 3-parametric interpolation Macdonald polynomials.

Proposition 3.2. *The RHS of* (3.1) *is an element of* $\Lambda_{t,s}$ *of degree* $\le |\mu|$*.*

In the proof it will be convenient to consider the following analogs of the algebra $\Lambda_{t,s}$. By

$$\Lambda^{\pm\pm}_{t,s}\subset \Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}]$$

denote the subalgebra of polynomials that are (anti)-invariant with respect to permutation of x^*_i and (anti)-invariant with respect to transformations

$$x^*_i \leftrightarrow (x^*_i)^{-1}\,.$$

In particular, $\Lambda^{++}_{t,s}=\Lambda_{t,s}$. We shall also write

$$\Lambda^{\pm\pm}_{t,s}[x]$$

to stress the dependence on variables $x_1,\dots,x_n$.

Proof of proposition 3.2. By analytic continuation we can assume that

$$t=q^\theta,\quad \theta=2k+1,\quad k=0,1,2,\dots\,.$$

In this case $\Pi(x;y;q,t)$ and $D(x;q,t)$ belong to $\Bbbk[x_1^{\pm1},\dots,x_n^{\pm1}]$.

Note that

$$\Pi(x^*;y^*;q,t)=0$$

if

$$y_i = qx_i, q^2x_i,\dots,tq^{-1}x_i$$

or if

$$y_i = q^{-1}x_{i+1}, q^{-2}x_{i+1},\dots,qt^{-1}x_{i+1}\,.$$

Therefore, for any i, we can change the limits of integration in (3.1) as follows

$$\int^{qx_i} d_q y_i\Big(\dots\Big) = \int^{q^{r+1}x_i} d_q y_i\Big(\dots\Big),\quad r,s=1,\dots,2k\,. \tag{3.2}$$

In particular, we can replace our integration by the following

$$\int_{y \preccurlyeq x} \Big(\dots \Big) = \int_{(q/t)^{1/2} x_2}^{(qt)^{1/2} x_1} \cdots \int_{(q/t)^{1/2} x_n}^{(qt)^{1/2} x_{n-1}} \Big(\dots \Big) .$$

Denote by $f(x,y)$ the polynomial

$$f(x,y) = V(y^*; q,t)\, \Pi(x^*; y^*; q,t)\, P^*_\mu(y; q,t, st^{1/2}) .$$

We have

$$f(x,y) \in \Lambda_{t,s}[x], \quad f(x,y) \in \Lambda^{--}_{t,st^{1/2}}[y] .$$

In other words, $f(x,y)$ is a W-antiinvariant polynomial in variables y_i^* with coefficients in $\Lambda_{t,s}[x]$.

The following determinants $\mathcal{D}_l$ form a linear basis in $\Lambda^{--}_{t,st^{1/2}}[y]$

$$\mathcal{D}_l(y) = \det \begin{pmatrix} & \vdots & \\ \dots & (y_i^*)^{l_j} - (y_i^*)^{-l_j} & \dots \\ & \vdots & \end{pmatrix}_{1 \le i,j \le n-1} ,$$

where

$$l_1 > l_2 > \dots > l_{n_1} > 0 .$$

Using (A.2) we can evaluate the integral

$$\int_{(q/t)^{1/2} x_2}^{(qt)^{1/2} x_1} \cdots \int_{(q/t)^{1/2} x_n}^{(qt)^{1/2} x_{n-1}} \mathcal{D}_l(y) \, \frac{d_q y}{y}$$

explicitly and obtain, up to a constant factor,

$$\det \begin{pmatrix} & \vdots & \\ \dots & (x_i^*)^{l_j} + (x_i^*)^{-l_j} - (x_{i+1}^*)^{l_j} - (x_{i+1}^*)^{-l_j} & \dots \\ & \vdots & \end{pmatrix}_{1 \le i,j \le n-1} .$$

This $(n-1) \times (n-1)$ determinant is equal to the following $n \times n$ determinant

$$\det \begin{pmatrix} (x_1^*)^{l_1} + (x_1^*)^{-l_1} & \dots & (x_1^*)^{l_{n-1}} + (x_1^*)^{-l_{n-1}} & 1 \\ \vdots & & \vdots & \vdots \\ (x_n^*)^{l_1} + (x_n^*)^{-l_1} & \dots & (x_n^*)^{l_{n-1}} + (x_n^*)^{-l_{n-1}} & 1 \end{pmatrix} .$$

Note that the result is an element of $\Lambda^{-+}_{t,s}[x]$.

Denote by I the integral

$$I = \int \quad \beta_{q,t}(d_q y)\, P^*_\mu(y; q,t, st^{1/2}) .$$

By the above considerations, we have

$$I \in \Lambda^{-+}_{t,s}[x] \,.$$

Since (A.1) is always divisible by $(u-v)$ it follows from (3.2) that I is divisible, for example, by

$$(qt^{-1}x_1^* - x_2^*)\dots(q^{-1}x_1^* - x_2^*)(x_1^* - x_2^*)(qx_1^* - x_2^*)\dots(tq^{-1}x_1^* - x_2^*)\,.$$

By symmetry, I is divisible by $D(x^*;q,t)$, and moreover

$$\frac{I}{D(x^*;q,t)} \in \Lambda_{t,s}[x]\,.$$

Finally, observe that

$$\begin{aligned} \deg V(y^*;q,t) \quad &= n(n-1)/2\,,\\ \deg \Pi(x^*;y^*;q,t) &= 2kn(n-1)\,,\\ \deg D(x^*;q,t) \quad &= (4k+1)n(n-1)/2\,. \end{aligned}$$

Therefore, the degree of the RHS of (3.1) is less or equal to $|\mu|$. This concludes the proof. □

Proof of the theorem 3.1. Again, we can assume $\theta = 2k+1$, $k = 0,1,2,\dots$.

One checks the vanishing of the RHS of (3.1) at the points $x = q^\lambda$, $\mu \not\subset \lambda$ by precisely the same argument as in the A-series case, see [Ok3], §4. If $x = q^\lambda$ and θ is an odd natural number, then the q-integral is just a finite sum all summand of which vanish. Since the denominator $D(x^*;q,t)$ does not vanish at $x = q^\lambda$, the equality (3.1) holds up to a constant factor.

To show that this factor equals 1, one can either compute the RHS at $x = q^\mu$ (there will be only one non-vanishing summand in the integral), or one can consider the highest degree term of (3.1). This highest term will be computed explicitly in the next subsection. This will conclude the proof of the theorem. □

4. Highest degree term

Denote by

$$P_\mu(x_1,\dots,x_n;q,t)$$

the A-series Macdonald polynomial with parameters q and t.

Theorem 4.1 (Highest degree term).

$$P^*_\mu(x_1,\dots,x_n;q,t,s) = P_\mu(x_1,x_2t^{-1},\dots,x_nt^{1-n};q,t) + \dots\,, \tag{4.1}$$

where dots stand for lower degree terms.

By definition, set

$$\begin{aligned} \widehat{V}(x_1,\dots,x_n) &= \det\left[x_i^{n-j}\right]_{1\le i,j\le n}\,,\\ \widehat{\Pi}(x_1,\dots,x_n;y_1,\dots,y_{n-1};q,t) &= \prod_{i,j}\frac{(qy_j/x_i)_\infty}{(ty_j/x_i)_\infty}\,,\\ \widehat{D}(x_1,\dots,x_n;q,t) &= \prod_{i<j}(x_i-x_j)\prod_{i\ne j}\frac{(qx_i/x_j)_\infty}{(tx_i/x_j)_\infty}\,,\\ \widehat{C}(\mu,n) &= \prod B_q(\mu_i+(n-i)\theta,\theta)\,, \end{aligned}$$

The hats mean that these products are related to the top homogeneous term of $P^*_\mu(x;q,t,s)$. Set also

$$\begin{aligned} \hat{x}_i &= x_i t^{1-i}\,, \quad i = 1,\dots,n\,, \\ \hat{y}_i &= y_i t^{-i}\,, \quad i = 1,\dots,n-1\,. \end{aligned}$$

We shall deduce the theorem 4.1 from the following q-integral representation [Ok3] of $P_\mu(x;q,t)$

$$P_\mu(x;q,t) = \frac{1}{\widehat{C}(\mu,n)} \frac{1}{\widehat{D}(x;q,t)} \int_{y \prec x} P_\mu(y;q,t)\, \widehat{V}(y)\, \widehat{\Pi}(x;y;q,t)\, d_q y\,,$$

where

$$\int_{y\prec x} d_q y = \int_{x_2}^{x_1} \cdots \int_{x_n}^{x_{n-1}} d_q y_1 \dots d_q y_{n-1}\,.$$

Proof. By proposition 2.1 it suffices to consider the case $\mu_n = 0$. In this case we shall use the q-integral representation of $P^*_\mu(x;q,t,s)$ and induction on n.

By analytic continuation, we can assume that

$$t = q^\theta, \quad \theta = 2k+1, \quad k = 0,1,2,\dots\,.$$

Remark that the highest degree (in variables u and v) term of the polynomial

$$\prod (1 - a u^{\pm 1} v^{\pm 1})$$

equals

$$-auv(1-au/v)(1-av/u) = a^2u^2(1-av/u)(1-a^{-1}v/u)\,.$$

Therefore

$$\Pi(x^*;y^*;q,t) = q^{k^2 n(n-1)} (t^{n-1}s)^{2kn(n-1)}\, \widehat{\Pi}(\hat{x};\hat{y};q,t) \prod_i \hat{x}_i^{2k(n-1)} + \dots$$

where dots stand for lower degree terms.

Similarly,

$$V(y^*) = (t^{n-1/2}s)^{n(n-1)/2}\, \widehat{V}(\hat{y}) \prod_i \hat{y}_i + \dots\,,$$

and

$$D(x^*;q,t) = q^{(2k+1)kn(n-1)/2} (t^{n-1}s)^{(4k+1)n(n-1)/2} \times \widehat{D}(\hat{x};q,t) \prod_i \hat{x}_i^{2k(n-1)} + \dots\,.$$

The powers of q, t, and s in the three above formulas combine to

$$\frac{q^{k^2 n(n-1)} (t^{n-1}s)^{2kn(n-1)} (t^{n-1/2}s)^{n(n-1)/2}}{q^{(2k+1)kn(n-1)/2} (t^{n-1}s)^{(4k+1)n(n-1)/2}} = q^{n(n-1)/4}$$

Note that

$$\frac{d_q y_i}{y_i} = \frac{d_q \hat{y}_i}{\hat{y}_i}$$

and

$$\int_{x_{i+1}}^{qx_i} \frac{d_q y_i}{y_i} (\dots) = \int_{x_{i+1}}^{tx_i} \frac{d_q y_i}{y_i} (\dots) = \int_{\hat{x}_{i+1}}^{\hat{x}_i} \frac{d_q \hat{y}_i}{\hat{y}_i} (\dots),$$

where the first equality is based on (3.2).

By inductive assumption (note the difference in the definition of $\hat{x}_i$ and $\hat{y}_i$) we have

$$P^*_\mu(y_1, \dots, y_{n-1}; q, t, s) = t^{|\mu|} P_\mu(\hat{y}_1, \dots, \hat{y}_{n-1}; q, t) + \dots .$$

Therefore, assuming that the equality (3.1) holds up to a constant factor c, we conclude

$$P^*_\mu(x_1, \dots, x_n; q, t, s) = c P_\mu(\hat{x}_1, \dots, \hat{x}_n; q, t) + \dots .$$

But then setting $x_n = 1$ and using proposition 2.1, inductive hypothesis and the stability of A_n-series Macdonald polynomials, we immediately find

$$c = 1 .$$

This concludes the proof of the theorem and also the proof of the theorem 3.1 of the previous section. $\square$

5. Combinatorial formula

Recall that the polynomials $P^*_\mu(x; q, t, s)$ form a linear basis in $\Lambda_{t,s}$.

Definition 5.1. Let the polynomials

$$\psi_{\mu,\nu}(u; n) \in \Bbbk[u^{\pm 1}]$$

be the coefficients in the following expansion

$$P^*_\mu(u, x_2, \dots, x_n; q, t, s) = \sum_\nu \psi_{\mu,\nu}(u; n)\, P^*_\nu(x_2, \dots, x_n; q, t, s) . \tag{5.1}$$

Write

$$\nu \prec \mu$$

if $\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \dots \geq \nu_{n-1} \geq \mu_n$.

Theorem 5.1 (Branching rule). *We have*

$$\psi_{\mu,\nu}(u; n) = \tag{5.2}$$

$$\psi_{\mu/\nu}\, t^{-|\nu|} \prod_{\square \in \mu/\nu} \left(u - q^{a'(\square)} t^{-l'(\square)} \right) \left(1 - \frac{1}{q^{a'(\square)} t^{2n-2-l'(\square)} s^2 u} \right),$$

provided $\nu \prec \mu$ and $\psi_{\mu,\nu}(u; n) = 0$ otherwise. Here $\psi_{\mu/\nu}$ are the same weights that appear in the branching rule for ordinary A-type Macdonald polynomials.

Explicit formulas for the weights $\psi_{\mu/\nu}$ are given in [M], VI.6.24 and Example VI.6.2.(b); the last formula is reproduced in (0.4). We will not use explicit formulas for $\psi_{\mu/\nu}$ in the proof of (5.2).

The branching rule immediately results in the following combinatorial formula for $P^*_\mu(x; q, t, s)$. Call a tableau T on a diagram μ a *reverse tableau* if its entries strictly decrease down the columns and weakly decrease in the rows. Denote by $T(s)$ the entry of T in the square $s \in \mu$.

Theorem 5.2 (Combinatorial formula). *We have*

$$P^*_\mu(x_1,\dots,x_n;q,t,s)=\sum_T \psi_T \prod_{s\in\mu} t^{1-T(s)}\times \left(x_{T(s)}-q^{a'(\square)}t^{-l'(\square)}\right)\left(1-\frac{1}{q^{a'(\square)}t^{2(n-T(s))-l'(\square)}s^2x_{T(s)}}\right), \tag{5.3}$$

where the sum is over all reverse tableaux on μ with entries in $\{1,\dots,n\}$.

Here

$$\psi_T\in\mathbb{C}(q,t)$$

is the same (q,t)-weight of a tableau which enters the combinatorial formula for ordinary Macdonald polynomials (see [M],§VI.7)

$$P_\mu(x;q,t)=\sum_T \psi_T \prod_{s\in\mu} x_{T(s)}\,.$$

Comparing this combinatorial formula with the combinatorial formula [Ok3] for the 2-parametric interpolation Macdonald polynomials $P^*_\mu(x;q,t)$ one obtains the following proposition, in which part (b) follows from (2.2).

Proposition 5.3.

$$\begin{aligned}
&\text{a)} && P^*_\mu(x;q,t,s)\to P^*_\mu(x;q,t)\,, && s\to\infty\,,\\
&\text{b)} && s^{-2|\mu|}P^*_\mu(x;q,t,s)\to t^{(2-2n)|\mu|}P^*_\mu(1/x;1/q,1/t)\,, && s\to 0\,.
\end{aligned}$$

In the proof of the branching rule we shall induct on n and use the following corollary of this theorem

Corollary 5.4 of theorem 5.1. *For all $r=1,\dots,n$ we have*

$$P^*_\mu(x_1,\dots,x_r,q^{\mu_{r+1}},\dots,q^{\mu_n};q,t,s)=c\,x_1^{\mu_1}\dots x_r^{\mu_r}+\dots\,, \tag{5.4}$$

where c is a non-zero constant and dots stand for lower monomials in lexicographic order.

Proof. Given any partition ν set

$${}^{(i)}\nu=(\nu_i,\nu_{i+1},\dots)\,.$$

Set also

$${}^{\backprime}\mu={}^{(1)}\mu\,.$$

Consider the leading term of $P^*_\mu(x;q,t,s)$ as of a polynomial in x_1. Then (5.2) asserts that this leading term equals

$$x_1^{\mu_1}t^{-|{}^{\backprime}\mu|}P^*_{{}^{\backprime}\mu}(x_2,\dots,x_n;q,t,s)\,, \tag{5.5}$$

because $\psi_{\mu/{}^{\backprime}\mu}=1$. Using (5.2) again one obtains the leading term of (5.5) in x_2 and so on. Finally, observe that

$$P^*_{{}^{(i)}\mu}(q^{{}^{(i)}\mu})\neq 0$$

for all i. $\square$

We shall also need the following elementary

Lemma 5.5. *Let λ be a partition. Then*

$$P^*_\mu(x_1,\dots,x_{i-1},q^{\lambda_i},\dots,q^{\lambda_n};q,t,s)=0 \quad \textit{unless} \quad {}^{(i)}\mu\subset{}^{(i)}\lambda\,.$$

Proof. By definition of $P^*_\mu(x;q,t,s)$, this polynomial vanishes at all points

$$x_1=q^{\nu_1},\dots,x_{i-1}=q^{\nu_{i-1}}\,,$$

where

$$\nu_1\ge\dots\ge\nu_{i-1}\ge\lambda_i$$

are arbitrary integers, provided ${}^{(i)}\mu\not\subset{}^{(i)}\lambda$. □

Proof of theorem 5.1. Induct on n. The case $n=1$ is clear.

Fix some i. Show that if $\nu_i<\mu_i$ then

$$\psi_{\mu,\nu}(u;n)=0\,,\quad u=q^{\nu_i}t^{1-i},\dots,q^{\mu_i-1}t^{1-i}\,. \tag{5.6}$$

We shall prove (5.6) by induction on the partition ${}^{(i)}\nu$, that is we shall deduce (5.6) from the assumption that

$$\psi_{\mu,\eta}(u;n)=0\,,\quad u=q^{\eta_i}t^{1-i},\dots,q^{\mu_i-1}t^{1-i}\,,$$

for all η such that

$${}^{(i)}\eta\subsetneqq{}^{(i)}\nu\,.$$

Suppose

$$u=q^kt^{1-i},\quad k=\nu_i,\dots,\mu_i-1\,,$$

and consider the expansion

$$\begin{aligned}P^*_\mu(q^kt^{1-i},x_2,\dots,x_i,q^{\nu_i},\dots,q^{\nu_{n-1}};q,t,s)=\\ =\sum_\eta\psi_{\mu,\eta}(q^kt^{1-i};n)\,P^*_\eta(x_2,\dots,x_i,q^{\nu_i},\dots,q^{\nu_{n-1}};q,t,s)\,.\end{aligned} \tag{5.7}$$

By the lemma only summands satisfying

$${}^{(i)}\eta\subset{}^{(i)}\nu$$

are nonzero. On the other hand, if

$${}^{(i)}\eta\subsetneqq{}^{(i)}\nu$$

then in particular $\eta_i\le\nu_i$ and by our assumption $\psi_{\mu,\eta}(q^kt^{1-i};n)=0$. Therefore only summands with

$${}^{(i)}\eta={}^{(i)}\nu$$

enter the sum.

By the corollary 5.4 applied to polynomial

$$P^*_\eta(x_2,\dots,x_i,q^{\nu_i},\dots,q^{\nu_{n-1}};q,t,s)$$

in $n-1$ variable, each summand in (5.7) has the following form

$$c_\eta \psi_{\mu,\eta}(q^k t^{1-i}; n) \left(x_2^{\eta_1} \dots x_i^{\eta_{i-1}} + \dots\right), \tag{5.8}$$

where c_η is a nonzero factor and dots stand for lower monomials in lexicographic order.

On the other hand, by the symmetry of $P^*_\mu(x; q, t, s)$ we have

$$P^*_\mu(q^k t^{1-i}, x_2, \dots, x_i, q^{\nu_i}, \dots, q^{\nu_{n-1}}; q, t, s) =$$
$$= P^*_\mu(x_2/t, \dots, x_i/t, q^k, q^{\nu_i}, \dots, q^{\nu_{n-1}}; q, t, s) = 0$$

by lemma 5.5, provided $\nu_i \le k < \mu_i$. Therefore the polynomial (5.7) is identically zero. By virtue of (5.8) it is impossible unless

$$\psi_{\mu,\eta}(q^k t^{1-i}; n) = 0$$

for all η such that ${}^{(i)}\eta = {}^{(i)}\nu$. This proves (5.6).

Since $\psi_{\mu,\nu}(u; n)$ is invariant under the transformation

$$u \mapsto \frac{1}{t^{2n-2} s^2 u}$$

it vanishes also at the points

$$\frac{1}{t^{2n-2} s^2 u} = q^{\mu_i - 1} t^{1-i}, \dots, q^{\nu_i} t^{1-i},$$

for all i such that $\nu_i < \mu_i$.

Now show that

$$\psi_{\mu,\nu}(u; n) = 0$$

if $\nu \not\subset \mu$. Suppose that

$$\nu \not\subset \mu$$

and $\psi_{\mu,\nu}(u; n)$ is not identically zero. Since we know some zeroes of the polynomial $\psi_{\mu,\nu}(u; n)$ we have

$$\deg \psi_{\mu,\nu}(u; n) \ge \sum_i \max\{\mu_i - \nu_i, 0\}.$$

Hence

$$\deg \psi_{\mu,\nu}(u; n)\, P^*_\nu(x_2, \dots, x_n; q, t, s) \ge \sum_i \max\{\mu_i, \nu_i\} > |\mu|.$$

Therefore such a summand cannot occur in the expansion (5.1).

Thus we can assume that

$$\nu \subset \mu.$$

Since we know $2|\mu/\nu|$ distinct zeroes of $\psi_{\mu,\nu}(u; n)$ and again

$$\deg \psi_{\mu,\nu}(u; n) \le |\mu/\nu|$$

this polynomial should up to a scalar factor equal

$$\prod_{\square \in \mu/\nu} \left(u - q^{a'(\square)} t^{-l'(\square)}\right) \left(1 - \frac{1}{q^{a'(\square)} t^{2n-2-l'(\square)} s^2 u}\right).$$

This factor equals $\psi_{\mu/\nu}\, t^{-|\nu|}$ because the highest degree term of

$$P^*_\mu(x_1, \dots, x_n; q, t, s)$$

is the A-series Macdonald polynomial

$$P_\mu(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}; q, t).$$

This concludes the proof. $\square$

6. Pieri formulas and Cauchy identity

Definition 6.1. Let $\psi'_{\lambda,\mu}(u,n)$ be the coefficients in the following expansion

$$\prod_{i=1}^{n} \left(u + x_i t^{1-i}\right) \left(1 + \frac{1}{s^2 t^{2n-i-1} u x_i}\right) P^*_\mu(x;q,t,s) = \\ = \sum_\lambda \psi'_{\lambda,\mu}(u,n)\, P^*_\lambda(x;q,t,s)\,. \tag{6.1}$$

It is clear that $\psi'_{\lambda,\mu}(u,n)$ is a polynomial in u and u^{-1} invariant under the transformation

$$u \mapsto \frac{1}{s^2 t^{2n-2} u}\,.$$

Denote by $\mu + \bar{1}$ the following partition

$$\mu + \bar{1} = (\mu_1 + 1, \dots, \mu_n + 1)\,.$$

Denote by $\psi'_{\lambda/\mu}$ the coefficients of the Pieri formula for ordinary A_n-type Macdonald polynomials (see [M], VI.6)

$$\prod_{i=1}^{n} (u + x_i)\; P_\mu(x;q,t) = \sum_\lambda \psi'_{\lambda/\mu}\, u^{|\mu+\bar{1}/\lambda|}\, P_\lambda(x;q,t)\,. \tag{6.2}$$

Theorem 6.1 (Pieri formulas). *We have*

$$\psi'_{\lambda,\mu}(u,n) = \\ \psi'_{\lambda/\mu} \prod_{\square \in \mu+\bar{1}/\lambda} \left(u + q^{a'(\square)} t^{-l'(\square)}\right) \left(1 + \frac{1}{s^2 q^{a'(\square)} t^{2(n-1)-l'(\square)} u}\right), \tag{6.3}$$

provided

$$\mu \subset \lambda \subset \mu + \bar{1}\,,$$

and $\psi'_{\lambda,\mu}(u,n) = 0$ *otherwise.*

Pieri formulas for the interpolation Jack polynomials were considered in [KS].

Proof. First show that

$$\psi'_{\lambda,\mu}(u,n) = 0 \quad \text{if} \quad \mu \not\subset \lambda \tag{6.4}$$

by induction on λ. Assume that

$$\psi'_{\eta,\mu}(u,n) = 0$$

for all partitions η such that $\eta \subsetneq \lambda$. Then evaluation of (6.1) at $x = q^\lambda$ gives (6.4).

From now on we suppose

$$\mu \subset \lambda$$

Set $\mu'_0 = n$ and suppose that $\lambda'_k < \mu'_{k-1}$ for some $k \geq 1$. Show that

$$\psi'_{\lambda,\mu}(-q^{k-1}t^{1-i}, n) = 0\,, \quad i = \lambda'_k + 1, \dots, \mu'_{k+1}\,. \tag{6.5}$$

Again, we assume that

$$\psi'_{\eta,\mu}(-q^{k-1}t^{1-i}, n) = 0\,, \quad i = \eta'_k + 1, \dots, \mu'_{k+1}\,.$$

for all partitions η such that

$$\eta \subsetneqq \lambda$$

and evaluate (6.1) at

$$x = q^\lambda, \quad u = -q^{k-1}t^{1-i}, \quad i = \lambda'_k + 1, \dots, \mu'_{k+1}\,.$$

This gives (6.5).

Since $\psi'_{\lambda,\mu}(u,n)$ is invariant with respect to the transformation

$$u \mapsto \frac{1}{s^2 t^{2n-2} u}\,,$$

we have

$$\psi'_{\lambda,\mu}\left(-\frac{1}{s^2 q^{k-1} t^{2n-i-1}}, n\right) = 0\,, \quad i = \lambda'_k + 1, \dots, \mu'_{k+1}\,. \tag{6.6}$$

Now prove that

$$\psi'_{\lambda,\mu}(u,n) = 0 \quad \text{if} \quad \lambda \not\subset \mu + \bar{1}\,. \tag{6.7}$$

Suppose that $\psi'_{\lambda,\mu}(u,n) \neq 0$. Then by (6.5) and (6.6)

$$\deg \psi'_{\lambda,\mu}(u,n) \geq \#\{i \,|\, \mu_i = \lambda_i\}\,.$$

If $\lambda \not\subset \mu + \bar{1}$ then

$$\deg \psi'_{\lambda,\mu}(u,n)\, P^*_\lambda(x;q,t,s) \geq \sum_i \max\{\mu_i + 1, \lambda_i\} > |\mu| + n\,,$$

but then such a summand cannot enter the RHS of (6.1).

If $\lambda \subset \mu + \bar{1}$ then by the same reason

$$\deg \psi'_{\lambda,\mu}(u,n) \leq |\mu + \bar{1}/\lambda|\,.$$

By (6.5) and (6.6) we have

$$\psi'_{\lambda,\mu}(u,n) = \\ \text{const} \prod_{\square \in \mu+\bar{1}/\lambda} \left(u + q^{a'(\square)} t^{-l'(\square)}\right) \left(1 + \frac{1}{s^2 q^{a'(\square)} t^{2(n-1)-l'(\square)} u}\right).$$

The constant factor is determined by (4.1) and (6.2). This concludes the proof of the theorem. $\square$

The similarity of the proof of Pieri formulas and of the combinatorial formula is not accidental. In fact, these two theorems are equivalent to each other by virtue of the following theorem.

Theorem 6.2 (Cauchy identity).

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\left(x_i t^{n-i}-y_j q^{m-j}\right)\left(1-\frac{1}{s^2 q^{m-j} t^{n-i} x_i y_j}\right)= \tag{6.8}$$

$$=\sum_{\mu\subset(m^n)}(-1)^{|\widetilde{\mu}|}t^{(n-1)|\mu|}q^{(m-1)|\widetilde{\mu}|}\,P^*_\mu(x;q,t,s)\,P^*_{\widetilde{\mu}}(y;t,q,s)\,,$$

where

$$\widetilde{\mu}=(n-\mu'_m,\dots,n-\mu'_1)\,.$$

It follows from (6.8) that the branching rule in variables x is equivalent to Pieri formula in variables y and vice versa.

Lemma 6.3. *Suppose $\mu\subset(m^n)$ and λ has at most m parts. Then*

$$\prod_{i=1}^{n}\prod_{j=1}^{m}\left(q^{\mu_i}t^{n-i}-t^{\lambda_j}q^{m-j}\right)=0 \tag{6.9}$$

unless $\widetilde{\mu}\subset\lambda$.

Proof of the lemma. Suppose this product does not vanish. Then

$$\lambda_{m-\mu_i}\neq n-i\,,\quad i=1,\dots,n\,.$$

For $i=\mu'_1+1,\dots,n$ we obtain

$$\lambda_m\neq 0,1,\dots,n-\mu'_1-1\,.$$

Therefore

$$\lambda_m\geq n-\mu'_1\,.$$

For $i=\mu'_2+1,\dots,\mu'_1$ we obtain

$$\lambda_{m-1}\neq n-\mu'_1,\dots,n-\mu'_2-1\,.$$

Since $\lambda_{m-1}\geq\lambda_m\geq n-\mu'_1$ we obtain

$$\lambda_{m-1}\geq n-\mu'_2\,.$$

In the same way we obtain

$$\lambda_{m-2}\geq n-\mu'_3,\dots,\lambda_1\geq n-\mu'_m\,,$$

which means $\widetilde{\mu}\subset\lambda$. $\square$

Proof of the theorem. Denote by $f(x;y)$ the product in the LHS of (6.8). Observe that

$$f(x;y)\in\Lambda_{t,s}[x]\quad\text{and}\quad f(x;y)\in\Lambda_{q,s}[y]\,.$$

Expand $f(x;y)$ in polynomials $P^*_\mu(x;q,t,s)$

$$f(x;y)=\sum P^*_\mu(x;q,t,s)P^?_\mu(y)$$

where $P^?_\mu(y)$ are certain elements of $\Lambda_{q,s}[y]$ of degree

$$\deg P^?_\mu(y) \le |\widetilde{\mu}|\,.$$

Since m is the highest exponent of x_1 in $f(x;y)$ only partitions μ satisfying

$$\mu \subset (m^n)$$

enter the expansion of $f(x;y)$. We have to prove that

$$P^?_\mu(y) = (-1)^{|\widetilde{\mu}|} t^{(n-1)|\mu|} q^{(m-1)|\widetilde{\mu}|}\, P^*_{\widetilde{\mu}}(y;t,q,s)\,. \tag{6.10}$$

Induct on $|\mu|$. Assume that

$$P^?_\eta(y) = (-1)^{|\widetilde{\eta}|} t^{(n-1)|\eta|} q^{(m-1)|\widetilde{\eta}|}\, P^*_{\widetilde{\eta}}(y;t,q,s)\,,$$

for all partitions η such that

$$\eta \subsetneq \mu$$

Then we have

$$f(q^\mu;y) = P^*_\mu(q^\mu;q,t,s)P^?_\mu(y) + \sum_{\eta \subsetneq \mu} P^*_\eta(q^\mu;q,t,s)P^*_{\widetilde{\eta}}(y;t,q,s)\,.$$

Let λ range over all diagrams with $\le |\widetilde{\mu}|$ squares. Then

$$P^*_{\widetilde{\eta}}(t^\lambda;t,q,s) = 0$$

because $|\widetilde{\eta}| > |\widetilde{\mu}|$. By the lemma

$$f(q^\mu;t^\lambda) = 0$$

unless $\lambda = \widetilde{\mu}$. Therefore

$$P^?_\mu(t^\lambda) = 0$$

unless $\lambda = \widetilde{\mu}$. This proves (6.10) up to a scalar factor. This factor is determined by (4.1) and the following formula (6.11) for ordinary Macdonald polynomials which we recall. $\square$

Proposition 6.4.

$$\prod_{i=1}^n \prod_{j=1}^m (x_i - y_j) = \sum_{\mu \subset (m^n)} (-1)^{|\widetilde{\mu}|} P_\mu(x;q,t)\, P_{\widetilde{\mu}}(y;t,q)\,. \tag{6.11}$$

Proof. The identity (5.4) in [M], Ch. VI, reads

$$\prod_{i=1}^n \prod_{j=1}^m (1 + x_i y_j) = \sum_{\mu \subset (m^n)} P_\mu(x;q,t)\, P_{\mu'}(y;t,q)\,.$$

Therefore (6.11) is equivalent to

$$P_{\widetilde{\mu}}(y;t,q) = \left(\prod_i y_i^n\right) P_{\mu'}(1/y;t,q)\,.$$

It is clear that the RHS of the above equality is a polynomial in y. One easily checks that it is an eigenfunction of the Macdonald q-difference operator D_n^1 defined in [M], formula (VI.3.4). $\square$

7. Binomial formula for Koornwinder polynomials

The Koornwinder polynomials [K]

$$P_\lambda(x;q,t,a_1,\dots,a_4)\in\mathbb{C}(q,t,a_1,\dots,a_4)[x_1^{\pm1},\dots,x_n^{\pm1}]$$

are W-invariant and depend on six parameters

$$q,t,a_1,a_2,a_3,a_4\,.$$

These polynomials are orthogonal on the torus

$$|x_i|=1,\quad i=1,\dots,n$$

with respect to following measure

$$\frac{1}{(2\pi i)^n}\,\Delta(x)\,\frac{dx}{x}\,,$$

where

$$\Delta(x)=\prod_{i<j}\frac{(x_i^{\pm1}x_j^{\pm1})_\infty}{(tx_i^{\pm1}x_j^{\pm1})_\infty}\prod_i\frac{(x_i^{\pm1},-x_i^{\pm1},q^{1/2}x_i^{\pm1},-q^{1/2}x_i^{\pm1})_\infty}{(a_1x_i^{\pm1},-a_2x_i^{\pm1},q^{1/2}a_3x_i^{\pm1},-q^{1/2}a_4x_i^{\pm1})_\infty}\,.$$

Here, by definition,

$$(u,v,\dots,w)_\infty=(u)_\infty(v)_\infty\dots(w)_\infty$$

and

$$(u)_\infty=(1-u)(1-qu)(1-q^2u)\cdots\,.$$

These polynomials specialize to Macdonald polynomials for classical root systems, see [K] and also [D1], section 5. It is known ([D1], section 5.2) that the highest degree term of P_λ is the A-type Macdonald polynomial with parameters q and t.

The parameters $a_1,\dots,a_4$ are related to parameters a,b,c,d used by Koornwinder via

$$a=a_1,\quad b=-a_2,\quad c=q^{1/2}a_3,\quad d=-q^{1/2}a_4\,.$$

It is also convenient to introduce, following J. F. van Diejen, dual parameters $a_1',\dots,a_4'$ related to parameters $a_1,\dots,a_4$ by the following involution

$$\begin{pmatrix}\log a_1'\\ \log a_2'\\ \log a_3'\\ \log a_4'\end{pmatrix}=\frac12\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}\log a_1\\ \log a_2\\ \log a_3\\ \log a_4\end{pmatrix}.$$

In particular,

$$a_1'=\sqrt{a_1a_2a_3a_4}\,.$$

To simplify notation, we shall sometimes omit the six parameters and write simply $P_\lambda(x)$.

A q-difference operator whose eigenfunctions are $P_\lambda(x;q,t,a_1,\dots,a_4)$ was found by T. Koornwinder in [K]. In [D1] van Diejen found explicit q-difference operators D_k, $k=1,\dots,n$ such that

$$D_k P_\lambda = E_k(q^\lambda) P_\lambda$$

where

$$E_k \in \Lambda_{t,a'_1}$$

and the highest degree term of E_k is (up to a constant factor) the k-th elementary symmetric function in variables $q^{\lambda_i} t^{n-i}$.

The operators $D_1,\dots,D_n$ have the following crucial property. Define the vectors ρ and ρ' by

$$q^\rho = (t^{n-1}a'_1,\dots,ta'_1,a'_1)\,,$$
$$q^{\rho'} = (t^{n-1}a_1,\dots,ta_1,a_1)\,.$$

Suppose λ is a partition and consider the number

$$[D_k P_\lambda]\,(q^{\mu+\rho'})\,. \tag{7.1}$$

Since D_k is a q-difference operator this number is a combination of values of P_λ at several neighboring points with some coefficients which do not depend on λ. In fact, see Lemma 4.3 in [D2], only points of the form

$$x = q^{\nu+\rho'}\,,$$

where ν is a partition and

$$|\nu_i - \mu_i| \le 1, \quad \sum |\nu_i-\mu_i| \le k$$

contribute to (7.1). Moreover, the contribution of the point

$$\nu = (\mu_1+1,\dots,\mu_k+1,\mu_{k+1},\dots,\mu_n)$$

is nonzero.

In the context of Cherednik's double affine Hecke algebra such a property of q-difference operators can be proved in an abstract fashion, see [C1,2]. One can apply the Cherednik's algebra techniques to Koornwinder polynomials, see [M2,No1].

This property results in the following theorem

Theorem 7.1 (Binomial formula).

$$\frac{P_\lambda(xq^{\rho'};q,t,a_1,\dots,a_4)}{P_\lambda(q^{\rho'};q,t,a_1,\dots,a_4)} = \sum_{\mu\subset\lambda} t^{(n-1)|\mu|} a_1^{|\mu|} \frac{P^*_\mu(q^\lambda;q,t,a'_1) P^*_\mu(x;q,t,a_1)}{P^*_\mu(q^\mu;q,t,a'_1) P_\mu(q^{\rho'};q,t,a_1,\dots,a_4)}\,.$$

Recall that an explicit product expression for $P^*_\mu(q^\mu;q,t,a'_1)$ is given in definition 1.2 in section 1.

Proof. In the same way as in the A-series case (see the proof of the main theorem in [Ok4]) it follows from the above properties of the operators D_k that for any partition μ there exists an q-difference operator D_μ such that

$$[D_\mu P_\lambda]\,(q^{\rho'}) = P_\lambda(q^{\mu+\rho'})\,.$$

This operator D_μ is a polynomial in $D_1, \dots, D_n$ and

$$D_\mu\, P_\lambda = d_\mu(q^\lambda)\, P_\lambda\,,$$

where

$$d_\mu \in \Lambda_{t,a_1'}$$

and

$$\deg d_\mu \le |\mu|\,.$$

Consider the Newton interpolation of the function

$$f(x) = \frac{P_\lambda(xq^{\rho'}; q, t, a_1, \dots, a_4)}{P_\lambda(q^{\rho'}; q, t, a_1, \dots, a_4)}$$

with knots

$$x = q^\mu\,,$$

where μ ranges over partitions with at most n parts. This Newton interpolation has the form

$$f(x) = \sum_\mu b_{\mu,\lambda} P^*_\mu(x; q, t, a_1)\,,$$

where $b_{\mu,\lambda}$ is a linear combination of the values $f(q^\nu)$ for $\nu \subset \mu$. We have

$$f(q^\nu) = d_\nu(q^\lambda)\,.$$

Therefore,

$$b_{\mu,\lambda} = b_\mu(q^\lambda)$$

for some polynomial

$$b_\mu \in \Lambda_{t,a_1'}$$

of degree

$$\deg b_\mu \le |\mu|\,.$$

Since the highest degree term of $f(x)$ equals

$$\frac{t^{(n-1)|\lambda|} a_1^{|\lambda|}}{P_\lambda(q^{\rho'}; q, t, a_1, \dots, a_4)} P_\lambda(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}; q, t)\,,$$

where $P_\lambda(x; q, t)$ is the ordinary A-type Macdonald polynomial, and the highest degree term of $P^*_\mu(x; q, t, a_1)$ equals

$$P_\mu(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}; q, t)$$

we have

$$b_\mu(q^\lambda)=\begin{cases}0, & |\lambda|\le|\mu|,\lambda\neq\mu\,,\\ t^{(n-1)|\mu|}a_1^{|\mu|}/P_\mu(q^{\rho'};q,t,a_1,\dots,a_4), & \lambda=\mu\,.\end{cases}$$

Since $\deg b_\mu \le |\mu|$ the polynomial b_μ is completely determined by its values at the points q^λ, where $|\lambda|\le|\mu|$. Therefore $b_\mu(q^\lambda)$ is proportional to $P^*_\mu(q^\lambda;q,t,a'_1)$ and precisely

$$b_\mu(q^\lambda)=\frac{t^{(n-1)|\mu|}a_1^{|\mu|}}{P_\mu(q^{\rho'};q,t,a_1,\dots,a_4)}\,\frac{P^*_\mu(q^\lambda;q,t,a'_1)}{P^*_\mu(q^\mu;q,t,a'_1)}\,.$$

This concludes the proof. $\square$

The following important property of Koornwinder polynomials immediately follows from the binomial theorem

Theorem (J. P. van Diejen, [D2]). *If* $a'_1=a_1$ *then*

$$\frac{P_\lambda(q^{\nu+\rho'};q,t,a_1,\dots,a_4)}{P_\lambda(q^{\rho'};q,t,a_1,\dots,a_4)}=\frac{P_\nu(q^{\lambda+\rho};q,t,a_1,\dots,a_4)}{P_\nu(q^{\rho};q,t,a_1,\dots,a_4)}\,.$$

The general symmetry,

$$\frac{P_\lambda(q^{\nu+\rho'};q,t,a_1,\dots,a_4)}{P_\lambda(q^{\rho'};q,t,a_1,\dots,a_4)}=\frac{P_\nu(q^{\lambda+\rho};q,t,a'_1,\dots,a'_4)}{P_\nu(q^{\rho};q,t,a'_1,\dots,a'_4)} \tag{7.2}$$

conjectured in [D2], depends on a formula for $P_\lambda(q^{\rho'};q,t,a_1,\dots,a_4)$. The conjectural formula for this number (see formula (5.5) in [D2]) was proved in [D2] under the self-duality condition

$$a'_1=a_1\,.$$

According to the note added in proof to [D2], that formula (together with symmetry (7.2)) was proved recently by I. G. Macdonald (in preparation).

Appendix 1. q-Integrals

Recall that (see, for example, Ch. 1 of [GR])

$$\int_v^u z^l\,\frac{d_qz}{z}=\frac{1}{[l]_q}(u^l-v^l),\quad l\neq 0\,,\quad [l]_q=\frac{1-q^l}{1-q}\,. \tag{A.1}$$

If $u=q^kv$ for some $k\in\mathbb{N}$ then

$$\int_v^u f(z)\,\frac{d_qz}{z}=-(1-q)\sum_0^{k-1}f(vq^i)$$

for any polynomial $f(z)$ without constant term. For example, polynomials $f(z)\in\mathbb{C}[z^{\pm1}]$ which satisfy

do not have constant terms. Suppose an analytic function $f(z)$ is given by its Laurent series of the form

$$f(z) = \sum_1^\infty c_n (z^n - z^{-n}), \quad \delta^{-1} < |z| < \delta,$$

for some $\delta > 1$. By definition, set

$$\begin{aligned} \int_v^u f(z) \, \frac{d_q z}{z} &= (1-q) \sum_1^\infty c_n \left(\frac{1}{1-q^n} z^n - \frac{1}{1-q^{-n}} z^{-n} \right) \Bigg|_{z=v}^{z=u} \\ &= (1-q) \sum_1^\infty \frac{c_n}{q^{-n/2} - q^{n/2}} \left((zq^{-1/2})^n + (zq^{-1/2})^{-n} \right) \Bigg|_{z=v}^{z=u}, \end{aligned} \tag{A.2}$$

which converges if

$$|q|\delta^{-1} < |v|, |u| < \delta \,.$$

Note the following symmetries of (A.2)

$$\int_v^u f(z) \, \frac{d_q z}{z} = \int_v^{q/u} f(z) \, \frac{d_q z}{z} = \int_{q/v}^u f(z) \, \frac{d_q z}{z} = - \int_u^v f(z) \, \frac{d_q z}{z} \,.$$

Recall that the q-beta function is defined (see [GR],1.10) by

$$B_q(a,b) = (1-q) \frac{(q)_\infty (q^{a+b})_\infty}{(q^a)_\infty (q^b)_\infty} \,. \tag{A.3}$$

Appendix 2. Absence of rational q-difference equations

In this Appendix we show that the following polynomials $f(x)$ in one variable x

$$\begin{aligned} f_m(x) &= P^*_m(x; q, t, s) \\ &= (x-1)(x-q) \cdots (x - q^{m-1}) \left(1 - \frac{1}{s^2 x} \right) \cdots \left(1 - \frac{1}{s^2 q^{m-1} x} \right), \end{aligned}$$

where

$$m = 1, 2, \dots \,,$$

do not satisfy any q-difference equation of the form

$$\sum_{-d \le i \le d} a_i(x) f_m(q^i x) = E(m) f_m(x) \,, \quad d > 0 \,, \tag{A.4}$$

where $a_i(x)$ are rational functions in x and

$$a_d(x) \not\equiv 0 \quad \text{or} \quad a_{-d}(x) \not\equiv 0 \,.$$

Assume the polynomials $f_m(x)$ satisfy (A.4). Consider the points

$$x = q^{m-d}, \quad m = 2d, 2d+1$$

Only finitely many of them are poles of functions $a_i(x)$, $-d \le i \le d$. Evaluate (A.4) at the remaining points. We obtain

$$a_d(q^{m-d}) f_m(q^m) = 0\,.$$

Since $f_m(q^m) \neq 0$ we obtain

$$a_d(q^{m-d}) = 0$$

for infinitely many values of m. Therefore

$$a_d(x) \equiv 0\,.$$

Similarly, evaluation at the points

$$x = \frac{1}{s^2 q^{m-d}}, \quad m = 2d, 2d+1, \dots$$

gives

$$a_{-d}\left(\frac{1}{s^2 q^{m-d}}\right) = 0$$

for infinitely many values of m. Therefore

$$a_{-d}(x) \equiv 0\,.$$

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